ISRAEL JOURNAL OF MATHEMATICS **130** (2002), 145–155

ON A CONJECTURE OF FURSTENBERG

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ABSTRACT

The Hausdorff dimension of the set of numbers which can be written using digits $0, 1, t$ in base 3 is estimated. For every t irrational a lower bound $0.767\ldots$ is found.

1. Estimation of the Hausdorff dimension

1.1 STATEMENT OF THE PROBLEM. Consider an iterated function system Ψ_t given by three generators:

$$\psi_0(x) = \frac{x}{3}, \quad \psi_1(x) = \frac{x+1}{3}, \quad \psi_t(x) = \frac{x+t}{3}$$

where $t \in \mathbf{R}$ is a fixed parameter.

By [1], for every t there is a unique compact set Z_t which is invariant under Ψ_t and such that the orbit of any compact set under Ψ_t converges to Z_t in the

* Support from NSF grant DMS-0072312 and Penn State in the form of a sabbatical leave is acknowledged. This work was partly done when the author was visiting the Mathematical Institute of the Polish Academy of Sciences in Warsaw.
Received March 15, 2001

Hausdorff metric. An elementary intepretation of Z_t is as the set of numbers which can be represented by generally infinite expressions in base 3 which use digits $0, 1, t$.

In this paper we are proving the following:

THEOREM 1: *For every t irrational, $HD(Z_t) \geq 1 - \log(5/3)/2\log 3 > 0.767$.*

Since the Hausdorff dimension in an affine invariant, from now we will assume without loss of generality that $|t| \leq 1$. Theorem 1 will be derived from a technical Theorem 2 which is stated later.

Conjectures of Furstenberg: Let's quote three related conjectures of Furstenberg.

CONJECTURE 1: *For every t irrational, $HD(Z_t) = 1$.*

Let W be the limit set of the iterated function systems in $\mathbf{R}^2$ which is generated by $x \to x/3$, $x \to [x + (1,0)]/3$ and $x \to [x + (0,1)]/3$.

CONJECTURE 2: *For every t irrational and almost every $\beta \in \mathbf{R}$ the line $v = tu+\beta$ intersects W along a set with Hausdorff dimension 0.*

Let T denote the operator

$$Tf(x) = \frac{1}{3}\left[f(x) + f(x-1) + f(x-t)\right]$$

acting on the space of continuous functions with compact support.

CONJECTURE 3: *For every t irrational the spectral radius of the adjoint T^* is equal to 1.*

Historical remarks: Theorem 1 is a step towards proving Conjecture 1. Conjecture 1 was the subject of work by several authors. One should mention [3] where it was established that for almost every t, both in the topological and category sense, $HD(Z_t) = 1$, and that $|Z_t| = 0$ (Lebesgue measure) for every t irrational, see also [4]. In [2] a study of the continuity properties of the function $t \to HD(Z_t)$ was undertaken, while [6] contains numerical data mostly in support of Conjecture 1.

1.2 ENERGY ESTIMATE. Given a positive probabilistic measure μ on $\mathbf{R}$ and $\alpha \geq 0$, we define its *energy integral*

$$I_\alpha(\mu) := \int \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}.$$

For a Borel set $Z \subset \mathbf{R}$ consider the set $A(Z)$ which consists of those $\alpha \geq 0$ for which there exists a Radon measure μ_α supported on Z and $I_\alpha(\mu_\alpha) < \infty$. It is known that $HD(Z) = \sup A(Z)$, see [5]. Hence, each time we get a measure μ supported on Z and $I_\alpha(\mu) < \infty$, we have bounded the Hausdorff dimension of Z by α from below.

Natural measures: We will work with a concrete measure μ^t supported on Z_t. Consider a sequence of measures

$$\mu_0^t = \delta_0, \mu_1^t = \frac{1}{3}(\delta_0 + \delta_1 + \delta_t)$$

and

$$\mu_n^t = \mu_1^t * (\psi_{0*}\mu_1^t) * \cdots * (\psi_{0*}^{n-1}\mu_1^t)$$

for $n > 0$. The choice of equal weighting of the measures transferred by all generators was of course arbitrary. One easily verifies that

$$\mu_n^t = \mu_k^t * \phi_{0*}^k \mu_{n-k}^t$$

for $0 \leq k \leq n$. Hence, measures μ_n^t converge weakly to μ^t which is supported on the interval $[0, 1/2]$.

Estimates: Let us begin to estimate the energy integral. Let $0 < \alpha < 1$. For $n \geq 0$ denote $L_n := \{(x, y)\colon 2 \cdot 3^{-n} < |x - y| \leq 2 \cdot 3^{1-n}\}$.

$$I_\alpha(\mu^t) = \int \frac{d\mu^t(x)d\mu^t(y)}{|x-y|^\alpha} = \sum_{n=1}^{\infty} \int_{L_n} \frac{d\mu^t(x)d\mu^t(y)}{|x-y|^\alpha}.$$

Since $\mu^t = \mu_n^t * \psi_{0*}^n \mu^t$ and the support of $\psi_{0*}^n \mu^t$ is contained in $[-3^{-n}/2, 3^{-n}/2]$, we can write

$$\int_{L_n} \frac{d\mu^t(x)d\mu^t(y)}{|x-y|^\alpha} = \int_{L_n} |x-y|^{-\alpha} * h(x) * h(y) d\mu_n^t(x) d\mu_n^t(y)$$

where h is a non-negative function with total mass 1 and support contained in $[-3^{-2}/2, 3^{-n}/2]$. Because of that, for $(x, y) \in L_n$ we get

$$|x-y|^{-\alpha} * h(x) * h(y) \leq 3^{na}$$

and

$$I_\alpha(\mu^t) \leq \sum_{n=1}^{\infty} 3^{n\alpha} \int_{L_n} d\mu_n^t(x) d\mu_n^t(y). \tag{1}$$

Denote

$$
\begin{aligned}
s(n,\beta) &= 3^n \int \chi_{(-3^{-n}/2,3^{-n}/2]}(x-y-\beta)d\mu_n^t(x)d\mu_n^t(y) \\
&= 3^n \int \chi_{(-3^{-n}/2,3^{-n}/2]}(z-a)d(\mu_n^t * (\mu_n^t)')(z)
\end{aligned} \tag{2}
$$

where the apostrophe means the measure transported by the map $x \to -x$. Then

$$
3^{n\alpha} \int_{L_n} d\mu_n^t(x)d\mu_n^t(y) \le
$$

$$
\frac{s(n,\frac{5}{2}3^{-n}) + \cdots + s(n,\frac{11}{2}3^{-n}) + s(n,-\frac{5}{2}3^{-n}) + \cdots + s(n,-\frac{11}{2}3^{-n})}{3^{n(1-\alpha)}}.
$$

If we write $S_n = \sup_{\beta\in\mathbf{R}} s(n,\beta)$, then we get from estimate (1) that

$$
I_\alpha(\mu^t) \le 8\sum_{n=1}^{\infty} 3^{n(\alpha-1)} S_n. \tag{3}
$$

So the task is reduced to estimating the exponential rate of increase for S_n.

1.3 Projection measure. Consider a measure ν_1 in $\mathbf{R}^2$ defined by

$$
\nu_1 = \frac{1}{3}(\delta_{(0,0)} + \delta_{(0,1/3)} + \delta_{(1/3,0)}).
$$

If ψ denotes the homothety with scale $1/3$, then we define

$$
\nu_n = \nu_1 * (\psi_*\nu_1) \cdots * (\psi_*^{n-1}\nu_1).
$$

If $\pi_t\colon \mathbf{R}^2 \to \mathbf{R}$ denotes the linear projection given by $\pi_t(u,v) = tu + v$, then we have $\mu_n^t = \pi_{t*}\nu_n$. Hence

$$
\mu_n^t * (\mu_n^t)' = \pi_{t*}\left[\nu_1 * \nu_1' * \psi_*(\nu_1 * \nu_1') * \cdots * \psi_*^{n-1}(\nu_1 * \nu_1')\right].
$$

Measure $\nu_1 * (\nu_1)'$ is obtained explicitly and equals

$$
\frac{1}{9}\sum_{k,\ell=-1,0,1} b(k,\ell)\delta_{(k/3,\ell/3)} \tag{4}
$$

where $b(k,\ell) = 1$ if $k \ne \ell$, 3 if $k = \ell = 0$ and 0 otherwise. Function b extends to $\mathbf{Z}\times\mathbf{Z}$ by $b(k_1,\ell_1) = b(k,\ell)$ where $k,\ell = -1,0,1$ and $k-k_1, \ell-\ell_1 \in 3\mathbf{Z}$.

From the defining formula (2),

$$
\begin{aligned}
s(n,\beta) &= 3^n \int \chi_{(-3^{-n}/2,3^{-n}/2]}(z-\beta)d(\mu_n^t * (\mu_n^t)')(z) \\
&= 3^n \sum_{k,\ell\in\mathbf{Z}} (\nu_n * \nu_n')(k3^{-n}, \ell 3^{-n})\chi_{(-3^{-n}/2,3^{-n}/2]}(tk3^{-n} + \ell 3^{-n} - \beta).
\end{aligned}
$$

For every $k \in \mathbf{Z}$ and β, a non-zero contribution is obtained only when $\ell = \ell_{n,\beta}(k) := \langle 3^n(\beta - kt3^{-n})\rangle$ where $\langle x\rangle$ is the integer characterized by the condition $-1/2 < x - \langle x\rangle \le 1/2$. If we also introduce the notation $k_n(x) = \langle 3^n x\rangle$, then we can write

$$\begin{aligned} s(n,\beta) &= 3^n \sum_{k\in\mathbf{Z}} (\nu_n * \nu'_n)(k3^{-n}, \ell_{n,\beta}(k)3^{-n}) \\ &= 9^n \int_{-\infty}^{+\infty} (\nu_n * \nu'_n)\left(k_n(x)3^{-n}, k_n(\beta - tk_n(x))3^{-n}\right) dx. \end{aligned}$$

Define $b_i(u,v) = b(< 3^i u >, < 3^i v >)$. Then we can write for $-(3^n-1)/2 \le k, \ell \le (3^n-1)/2$ that

$$(\nu_n * \nu'_n)(k3^{-n}, \ell 3^{-n}) = 9^{-n} \prod_{i=1}^{n} b_i(k3^{-n}, \ell 3^{-n}).$$

If k, ℓ are outside that range, then $(\nu_n * \nu'_n)(k3^{-n}, \ell 3^{-n}) = 0$. Hence we can write

$$s(n,\beta) = \int_{-1/2}^{1/2} \prod_{i=1}^{n} B_i(x, \beta - tk_n(x))dx$$

where functions $B_i\colon T^2 \to \mathbf{R}$ are defined below.

Definition 1: If $(x,y) \in T^2$ and $i > 0$, then

$$B_i(x,y) := b(k_i(x), k_i(y)).$$

2. Averaging estimates

We will denote $T^1 := (-1/2, 1/2]$ and $T^2 := T^1 \times T^1$ and think of identifying pieces of the boundary so that tori are obtained. Let $\pi(x) := x'$ where $x' \in T^1$ and $x - x' \in \mathbf{Z}$. Recall that $k_n(x) = \langle 3^n x\rangle$.

2.1 Partitions related to base 3 expansions. It will be useful to think of the circle T^1 with the Lebesgue measure as a probabilistic space.

Definition 2: Say that an interval $I \subset T^1$ is a *basic interval* of order n, $n \ge 0$, if the transformation $x \to \pi(3^n x)$ maps I onto T^1 with degree 1.

For example, interval $(-\frac{1}{6}, \frac{1}{6}]$ is basic of order 1. Let $\mathcal{P}_n$ denote the partition of T^1 into basic intervals of order n.

Now let $\phi\colon \mathbf{R} \to \mathbf{R}$. For a positive integer n, define $\phi_n\colon T^1 \to T^1$ by

$$\phi_n(x) := \phi(3^{-n}k_n(x)). \tag{5}$$

So, ϕ_n is a $\mathcal{P}_n$-measurable approximation of ϕ.

LEMMA 1: *Let $q \geq 0$ and $n > 0$ and $\phi(x) = tx + t_0$. Then for every $x \in T^1$*

$$\phi_n(\pi(3^q x)) + T(x) - 3^q \phi_{n+q}(x)$$

is an integer, where $T(x) = k_q(x)t + (3^q - 1)t_0$.

Proof: The expression which is to be shown to yield an integer is measurable with respect to $\mathcal{P}_{n+q}$. It suffices to prove the claim for $x = (3^n J + j)3^{-n-q}$ with integers J and j ranging over $[-\frac{3^q-1}{2}, \frac{3^q-1}{2}]$ and $[-\frac{3^n-1}{2}, \frac{3^n-1}{2}]$, respectively. For x in such a form

$$3^q \phi_{n+q}(x) = 3^q(xt + t_0) = Jt + jt3^{-n} + 3^q t_0.$$

On the other hand,

$$\phi_n(\pi(3^q x)) = \phi_n(\pi(J + j3^{-n})) = \phi_n(j3^{-n}) = \pi(jt3^{-n} + t_0).$$

Finally, $k_q(x) = J$ and

$$T(x) = Jt + (3^q - 1)t_0$$

which implies the claim. ∎

Lemmas about circle rotations: Let $R_t: T^1 \to T^1$ be defined by $R_t(x) = \pi(x+t)$.

Definition 3: Define the set $U(t, K) \subset \mathbf{N}$ by the following requirement: $m \in U(t, K)$ if and only if for every $x \in T^1$ and every J which is a sub-arc of T^1 with length 3^{-m}, the set

$$\{R_t^p(x): p = 0, 1, \ldots, 3^m - 1\} \cap J$$

has no more than K elements.

Thus, for $m \in U(t, K)$ the first 3^m points of any orbit are uniformly spread out, in the sense that no "lumps" are formed.

LEMMA 2: *For every t irrational the set $U(t, 6)$ is infinite.*

Proof: Let q be a closest return time for the rotation $x \to x + t \bmod 1$. Then the orbit $x, \ldots, R_t^{q-1}(x)$ cuts the circle into pieces of two sizes and the shorter ones are never adjacent. Hence, any arc of length not exceeding $1/q$ may contain at most two points of the orbit. If m is chosen so that $3^{m-1} < q \leq 3^m$, the the orbit $x, \ldots, R_t^{3^m-1}(x)$ can be covered by three orbits of length q. Thus, no arc of length 3^{-m} contains more than 6 points. ∎

2.2 Averages along graphs.

Definition 4: Suppose that $F\colon T^2 \to \mathbf{R}$ and $g\colon T^1 \to \mathbf{R}$ are given. Then we can form a function $F_g\colon T^1 \to \mathbf{R}$ by the following formula: $F_g(x) = F(x, g(x))$.

The general type of the problem we will consider is as follows. We wish to average F_g along basic intervals, which corresponds to taking conditional expectations with respect to partitions $\mathcal{P}_n$. The problem is under what assumptions these averages can be estimated in terms of the average of F over T^2.

Proposition 1: *Consider $\phi(x) = tx + t_0$ and choose $N > 0$ and K so that $N \in U(t, K)$.*

For every $n \geq 0$ and every set $A \subset T^2$ which is measurable with respect to $\mathcal{P}_N \times \mathcal{P}_N$ we consider the function $F\colon T^2 \to \mathbf{R}$ given by

$$F^{(n)}(x,y) = \chi_A(\pi(3^{n+N}x), \pi(3^{n+N}y)).$$

Then,

$$E(F^{(n)}_{\phi_{n+2N}}|\mathcal{P}_n)(x) \leq K \int_{T^2} \chi_A d\lambda_2$$

for every $x \in T^1$, using the notation of Definition 4.

Proof: Choose an interval $I \in \mathcal{P}_n$. Observe first that without loss of generality $n = 0$. Indeed, for $n \geq 0$ the interval I can be parameterized by a variable $x' = \pi(3^n x)$ which runs over T^1. We get $\pi(3^N 3^n x) = \pi(3^N x')$ and, by Lemma 1,

$$\pi(3^{n+N}\phi_{n+2N}(x)) = \pi(3^N(\phi + T(x))_{2N}(x'))$$

with $T(x)$ constant and equal to $T(I)$ on I. Thus for every $x \in I$

$$E(F^{(n)}_{\phi_{n+2N}}|\mathcal{P}_n)(x) = E(F^{(0)}_{\phi_{2N}+T(I)})$$

which leads to the initial problem with $n = 0$ and t_0 increased by $T(I)$. Since the claim is supposed to be valid for every t_0, the reduction is complete.

We will write F for $F^{(0)}$ and ϕ for $\phi + T(I)$.

$$\begin{aligned} F_{\phi_{2N}}(x) &= \chi_A(\pi(3^N x), \pi(3^N \phi_{2N}(x))) \\ &= \chi_A\left[\pi(3^N x), \pi(\phi_N(\pi(3^N x)) + T(x))\right] \end{aligned}$$

by Lemma 1 used with $n = q = N$. If we write $x = J3^{-N} + j3^{-2N}$ with J, j both integers from the range $[-\frac{3^N-1}{2}, \frac{3^N-1}{2}]$, we get $\pi(3^N x) = j3^{-N}$ and

$T(x) = Jt + T_0$ where T_0 is a constant. We can then write

$$\begin{aligned} E(F_{\phi_{2N}}) &= E\left[\chi_A(\pi(3^N x), \pi(\phi_N(\pi(3^N x)) + T(x))\right] \\ &= 3^{-2N} \sum_{j=-(3^N-1)/2}^{(3^N-1)/2} \sum_{J=-(3^N-1)/2}^{(3^N-1)/2} \chi_A(j3^{-N}, \pi(Jt + \phi(j3^{-n}) + T_0)). \end{aligned}$$

For j fixed, points $\pi(Jt+\phi(j3^{-n})+T_0)$ form an orbit of the rotation R_t of length 3^N. By the hypothesis of the Lemma, each square of the partition $\mathcal{P}_N \times \mathcal{P}_N$ contains no more than K points in the form $(j3^{-N}, \pi(Jt + \phi(j3^{-n}) + T_0)))$. Hence,

$$3^{-2N} \sum_{J=-(3^N-1)/2}^{(3^N-1)/2} \sum_{j=-(3^N-1)/2}^{(3^N-1)/2} \chi_A(j3^{-N}, \pi(Jt+\phi(j3^{-n})+T_0)) \leq K \int_{T^2} \chi_A d\lambda_2. \quad \blacksquare$$

LEMMA 3: *For every $t \in \mathbf{R}$, $m \geq n \geq 0$, if $\phi(x) = tx$, $t_0 \in \mathbf{R}$, and $F: T^2 \to [0, \infty)$ is measurable with respect to $\mathcal{P}_n \times \mathcal{P}_n$, then*

$$F(\pi(x), \pi(\phi_m(x) + t_0)) \leq \sum_{2|i|<|t|+2} F(\pi(x), \pi(\phi_n(x) + t_0 + i3^{-n}))$$

where i runs through integer values only.

Proof: Estimate $|\phi_m(x) - \phi_n(x)| \leq |t|3^{-n}/2$. Thus, for every x we can choose $\tau(x)$ in the form $i3^{-n}$, where i is an integer and $-|t|/2 - 1 < i < |t|/2 + 1$ so that $\pi(\phi_n(x) + t_0 + \tau(x))$ and $\pi(\phi_m(x) + t_0)$ belong to the same element of $\mathcal{P}_n$, and so

$$F(\pi(x), \pi(\phi_m(x) + t_0)) = F(\pi(x), \pi(\phi_n(x) + t_0 + \tau(x))).$$

Now $\tau(x)$ only takes values in the set $i3^{-n}$ with $|i| < |t|/2 + 1$ and so the lemma follows. $\blacksquare$

PROPOSITION 2: *Let $t \in \mathbf{R}$, $K > 0$, $n \geq 0$ and $N \in U(t, K)$, see Definition 3. Denote $\phi(x) = tx$. Let $F: T^2 \to [0, \infty)$ be measurable with respect to $\mathcal{P}_n \times \mathcal{P}_n$. Suppose that for a fixed I and every $t_1 \in \mathbf{R}$,*

$$\int_{T^1} F_{\phi_n + t_1}(x) dx \leq I;$$

see Definition 4 for the explanation of the notation.

Now $G: T^2 \to [0, \infty)$ is measurable with respect to $\mathcal{P}_N \times \mathcal{P}_N$, $N > 0$. Define $\tilde{G}(x, y) = G(\pi(3^{n+N} x), \pi(3^{n+N} y))$.

Then, for every choice of t and K and every $t_0 \in \mathbf{R}$

$$\int_{T^1} F_{\phi_{n+2N}+t_0}(x)\, \tilde{G}_{\phi_{n+2N}+t_0}(x)dx \leq K(|t|+3)I \int_{T^2} G d\lambda_2.$$

Proof: Fix some $t_1 \in \mathbf{R}$. Function $F_{\phi_n+t_1}$ is measurable with respect to $\mathcal{P}_n$. By Proposition 1,

$$E(\tilde{G}_{\phi_{n+2N}+t_0}|\mathcal{P}_n)(x) \leq KI_G$$

for every x where $I_G := \int_{T^2} G d\lambda_2$. Now,

$$(6) \quad \begin{aligned}\int_{T^1} F_{\phi_n+t_1}(x)\tilde{G}_{\phi_{n+2N}+t_0}(x)dx &= \int_{T^1} E(F_{\phi_n+t_1}\, \tilde{G}_{\phi_{n+2N}+t_0}|\mathcal{P}_n)(x)dx \\ &\leq KI_G \int_{T^1} F_{\phi_n+t_1}(x)dx \leq KI_G I\end{aligned}$$

by the hypothesis of Proposition 2.

By Lemma 3

$$F_{\phi_{n+2N}+t_0}(x) = F(x, \pi(\phi_{n+2N}(x)+t_0)) \leq \sum_{j\in(-1-|t|/2,|t|/2+1)} F_{\phi_n+t_0+j3^{-n}}(x).$$

If we use estimate (6) for all $t_1 = t_0 + j3^{-n}$, we get

$$\int_{T^1} F_{\phi_{n+2N}+t_0}(x)\tilde{G}_{\phi_{n+2N}+t_0}(x)dx \leq KI_G(|t|+3)I. \quad \blacksquare$$

2.3 Averages of products.

Theorem 2: *Fix t irrational and let $\phi(x) = tx + t_0$. Then for every $\lambda > \sqrt{5/3}$ and $t_0 \in \mathbf{R}$ we have*

$$\lim_{m\to\infty}\left[\lambda^{-m}\int_{T^1}\prod_{i=1}^{m} B_i(x,\phi_m(x))dx\right] = 0.$$

Recall that functions B_i are given by Definition 1. Since

$$s(m,\beta) = \int_{T^1}\prod_{i=1}^{m} B_i(x,\phi_m(x))$$

with $\phi(x) = \beta - tx$, Theorem 2 implies that $\lambda^{-m}S_m \to 0$. By estimate (3), $I_\alpha(\mu^t) < \infty$ provided that $3^{1-\alpha} > \sqrt{5/3}$ and Theorem 1 follows.

Hölder estimate: From Lemma 2, see that $U(t,6)$ is infinite and choose $N \in U(t,6)$. Let $J_{k,0}$ denote the set of integers i which belong to $(2(j-1)N, (2j-1)N]$ for some $j = 1, \ldots, k$ and $J_{k,1}$ be the complement of $J_{k,0}$ in the set $1, 2, \ldots, 2kN$. Define $P_{k,0}(x,y) = \prod_{i \in J_{k,0}} B_i(x,y)$ and $P_{k,1}(x,y) = \prod_{i \in J_{k,1}} B_i(x,y)$. Then

$$\prod_{i=1}^{2kN} B_i(x, \phi_{2kN}(x)) = P_{k,0}(x, \phi_{2kN}(x)) P_{k,1}(x, \phi_{2kN}(x)).$$

Our approach is to apply the Hölder inequality to this product. It is easier to estimate the second norm of $P_{k,1}(x, (\phi+t_1)_{2kN})$ with $t_1 \in \mathbf{R}$. Using Proposition 2 with $n = 2(k-1)N + N$, $F := P^2_{k-1,1}$ and $G(x,y) = \prod_{i=1}^{N} B_i^2(x,y)$, we get

$$\|P_{k,1}(x, (\phi + t_1)_{2kN})\| \leq 6(|t| + 3) I \int_{T^2} G d\lambda_2$$

where I is an upper estimate for $\|P_{k-1,1}(x, (\phi+t_1)_{2(k-1)N}\|$ for any $t_1 \in \mathbf{R}$. Note that $\int_{T^2} G d\lambda_2 = (5/3)^N$ and hence one gets by induction starting with $P_{0,1} \equiv 1$ that

$$\|P_{k,1}(x, \phi_{2kN}(x))\|_2^2 \leq K_1^k (5/3)^{kN}.$$

The same method is used to estimate the second norm of

$$P_{k,0}(x, (\phi + t_1)_{(2k-1)N}(x)).$$

This time, the induction starts with $\|P_{1,0}(x, (\phi + t_1)_N(x))\|_2^2 \leq 3^N$ since 3^N is the maximum. Thus,

$$\|P_{k,0}(x, (\phi + t_1)_{(2k-1)N}(x))\|_2^2 \leq 3^N K_1^{k-1} (5/3)^{(k-1)N}.$$

Using Lemma 3 and applying the previous estimate for

$$t_1 = j 3^{-2(k-1)N}, \quad 2|j| < |t|,$$

we get

$$\begin{aligned} \|P_{k,0}(x, \phi_{2kN}(x))\|_2^2 &\leq (|t| + 3) \|P_{k,0}(x, \phi_{2(k-1)N}(x))\|_2^2 \\ &\leq (|t| + 1) 3^N K_1^{k-1} (5/3)^{2(k-1)N}. \end{aligned}$$

By Hölder's inequality,

$$\lambda^{-2kN} \int_{T^1} \prod_{i=1}^{2kN} B_i(x, \phi_{2kN}(x)) dx \leq \sqrt{3^N (|t| + 3)} \left[\frac{5}{3} \frac{\sqrt[N]{K_1}}{\lambda^2} \right]^{kN}.$$

If $\lambda > \sqrt{5/3}$ then N can be chosen so large that

$$\frac{5}{3}\frac{\sqrt[N]{K_1}}{\lambda^2} < 1.$$

Then

$$\lim_{k\to\infty}\left[\lambda^{-2kN}\int_{T^1}\prod_{i=1}^{2kN} B_i(x,\phi_{2kN}(x))dx\right] = 0. \tag{7}$$

Any $j > 0$ can be represented as $2k_jN + j_0$ with $j_0 < 2N$. Then

$$\prod_{i=1}^{j} B_i(x,\phi_j(x)) \leq 3^{j_0}\prod_{i=1}^{2k_jN} B_i(x,\phi_j(x)).$$

Again using Lemma 3 and the fact that we estimate

$$\int_{T^1}\prod_{i=1}^{2k_jN} B_i(x,\phi_j(x))dx \leq (|t|+1)\int_{T^1}\prod_{i=1}^{2k_jN} B_i(x,(\phi+t_1)_{2k_jN}(x))dx$$

where t_1 was chosen to attain the supremum of the integral on the right-hand side. Hence, for any $j > 0$,

$$\int_{T^1}\prod_{i=1}^{j} B_i(x,\phi_j(x))dx \leq 9^N(|t|+1)\int_{T^1}\prod_{i=1}^{2k_jN} B_i(x,(\phi+t_1)_{2k_jN}(x))dx$$

and Theorem 2 follows from this together with assertion (7). Notice that (7) holds for any t_0, in particular one can set $t_0 := t_0 + t_1$ in that estimate.

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